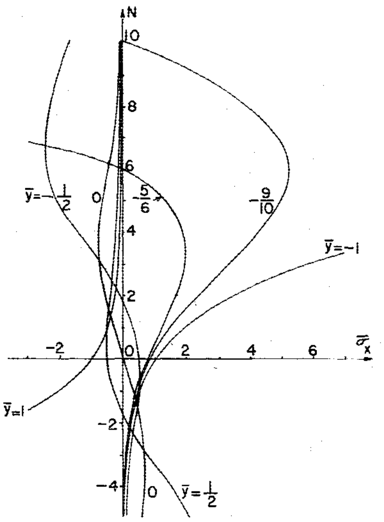

Fig. 3 Bending stress variation vs viscosity parameter for selected beam depths.

Discussions and Numerical Results

It should be noted that in Eq. (6), when the dimensionless viscosity parameter takes the value zero, one recovers the solution for a uniform viscous beam. This observation has considerable technical importance when one compares solutions for other values of N with the reference solution, since $N=0$ does not simply imply recovery of the elastic solution. Indeed, in some ways the use of N represents a mathematically useful description, but may suppress the physical importance of the viscosity variation. This also appears to be evident for some of the results presented in Ref. 2, which cannot be directly compared with the present solution due to differences in the problem analyzed. This is somewhat evident from Eq. (6), where it would appear that for values of $N>5$, a surface would essentially represent a fluid medium for response to any tractive loading.

Equations (11) are shown plotted in Fig. 2 ($\bar{x}=0$) for $N=-1, 0, 1$. It is seen that the $\bar{\sigma}_x$ curves for $N=1$ and $N=-1$ are symmetrical with respect to the origin, whereas the corresponding curves for $\bar{\tau}_{xy}$ are symmetrical with respect to the $\bar{\tau}_{xy}$ axis. Note that for positive N the upper face is stiffer and lower face softer, while the reverse is true for negative N.

It is also observed that the shear stress $\bar{\tau}_{xy}$ for the face layers remains independent of η_I/η_0, as long as $0.5 \leq \eta_I/\eta_0 \leq 1.5$ is satisfied. This is due to the fact that the shear stress is continuous at the interfaces, zero on the faces, and the faces are thin in comparison with the middle composite layer. However, $\bar{\sigma}_x$ is discontinuous at the interfaces and its value can be calculated from $\bar{\sigma}_x$ Eqs. (11a and 11c).

It is of some interest to note how changes in the variable viscosity parameter N may effect the position of the neutral axis.

For large N, $\bar{y}$ tends to one, and the neutral axis can be defined by the equations

$$\bar{y} = -\coth N + 1/N \quad \text{for all } N$$

$$\bar{y} = -1 + 1/N \quad \text{for } N \geq 3$$

As previously indicated, however, large values of N appear to be of little physical importance.

Finally, Fig. 3 shows the bending stress $\bar{\sigma}_x$ variation plotted vs N, with the beam depth plotted as a parameter. Except for the case $\bar{y}=\mp 1$, all curves are bounded. For a practical range of values, that is $|\bar{y}| < ½$, $|N| < 2.5$, the bending stress $\bar{\sigma}_x$ is bounded within a narrow ellipselike shape with maximum value of $|\bar{\sigma}_x| = 0.5$.

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High-Frequency Subsonic Flow Past a Pulsating Thin Airfoil

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Introduction

RECENTLY, much attention has been devoted to obtaining closed-form solutions to the linearized equations for the two-dimensional unsteady flow of an airfoil in a compressible stream. Low-frequency approximations for the subsonic lifting problem are treated in Amiet[1] and Kemp and Homicz,[2] and high-frequency approximations appear in Amiet[3] and Adamczyk.[4] For the high-frequency cases, an iterative procedure is used that calculates the leading- and trailing-edge flows separately.

For the lifting problem, approximate solutions are available for a wide range of upwash distributions. Although examples of practically important nonlifting problems are somewhat restricted, exact solutions are available as Green's function source distributions along the chordline of the airfoil. Amiet and Sears[5] found the low-frequency solution for a pulsating cylinder in subsonic flow. Horlock and Hawkings[6] studied the incompressible flow of a streamwise gust interacting with a symmetric airfoil.

It is the purpose of the present research to obtain high-frequency approximations for subsonic potential flow past a nonlifting airfoil. The general solution technique involves the asymptotic evaluation of the source integral for the perturbation velocity potential. It is applied to the case of flow past a pulsating airfoil.

Problem Formulation

Consider the two-dimensional unsteady subsonic potential flow of a uniform stream of speed U, Mach number M, and sound speed a past a thin nonlifting airfoil of chord $2l$ and thickness-to-chord ratio of $O(\epsilon)$. U and l are taken to be $O(1)$. An unsteady disturbance to the stream, harmonic in time with frequency ω and amplitude of $O(\epsilon)$, is caused by pulsation of the airfoil surface.

The velocity potential is taken as

$$\Phi(x,y,t) = Ux + \phi(x,y,t) \tag{1}$$

where ϕ, the perturbation velocity potential, is $O(\epsilon)$, and the Cartesian coordinate system is such that x is aligned with the stream and chord direction and the origin is at midchord. ϕ satisfies the following equation[7]

$$\begin{aligned}(1-M^2)\phi_{xx} + \phi_{yy} - 2Ma^{-1}\phi_{xt} - a^{-2}\phi_{tt} = a^{-2}[(\gamma-1)(2U\phi_x \\ +2\phi_t + \phi_x^2 + \phi_y^2)\nabla^2\phi/2 + 2(U\phi_x + \phi_x^2)\phi_{xx} + \phi_y^2\phi_{yy} \\ +2(U+\phi_x)\phi_y\phi_{yx} + 2(\phi_x\phi_{xt} + \phi_y\phi_{yt})]\end{aligned} \tag{2}$$

Received June 10, 1977; revision received Dec. 5, 1977; presented as Paper 78-226 at the AIAA 16th Aerospace Sciences Meeting, Huntsville, Ala., Jan. 16-18, 1978.

Index categories: Nonsteady Aerodynamics; Aeroacoustics; Subsonic Flow.

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where γ is the ratio of specific heats.

Consider a first-order (in ϵ) unsteady solution to Eq. (2) of the form $\epsilon\phi^{(1)}(x,y)\exp(i\omega t)$. It then satisfies the linearized equation

$$(1-M^2)\phi_{xx}^{(1)}+\phi_{yy}^{(1)}-2Mi\omega a^{-1}\phi_x^{(1)}+\omega^2 a^{-2}\phi^{(1)}=0 \quad (3)$$

This linearized equation is valid if the wavelength of the disturbance λ is large compared to the airfoil thickness; and if the frequency is also high, λ lies in the range

$$\epsilon \ll \lambda/l = 2\pi a/\omega l \ll 1 \quad (4)$$

Let the airfoil surface be described by

$$F(x,y,t)=y\mp\epsilon f(x,t)=0 \quad (5)$$

where the plus sign refers to the upper surface. The body boundary condition is

$$DF/Dt=F_t+\nabla\phi\cdot\nabla F=0 \text{ on } F=0 \quad (6)$$

In terms of ϕ, Eq. (6) becomes

$$\phi_y(x,\pm\epsilon f)=\pm\epsilon f_t\pm(U+\phi_x)\epsilon f_x \quad (7)$$

For the first-order solution Eq. (7) can be linearized and transferred to the chordline to yield

$$\phi_y^{(1)}(x,0\pm)=\pm w^{(1)}(x) \quad (8)$$

The solution to Eq. (3) with the boundary condition in Eq. (8) can be written as

$$\phi^{(1)}(x,y)=2\int_{-l}^{l} w^{(1)}(\xi)G(x-\xi,y)\,d\xi \quad (9)$$

where

$$G(x,y)=i(4\beta)^{-1}\exp(i\omega Mx/a\beta^2)H_0^{(2)}[\omega(x^2+\beta^2y^2)^{1/2}/a\beta^2] \quad (10)$$

and

$$\beta^2=1-M^2$$

Pulsating Airfoil

Consider the flow of a uniform stream past a pulsating airfoil whose surface is described by

$$f(x,t)=g(x)\exp(i\omega t) \quad (11)$$

where $g(x)$ and $g'(x)$ are functions of $O(1)$. This problem has application in acoustics and also is of interest theoretically, as will be seen by the nature of the solution. The first-order solution satisfies Eq. (3), and, for high frequency, the upwash is

$$w^{(1)}(x)=i\omega g(x) \quad (12)$$

From Eqs. (9) and (10), the perturbation velocity potential is

$$\phi^{(1)}(x,y)=-\omega(2\beta)^{-1}\int_{-l}^{l} g(\xi)\exp[i\omega M(x-\xi)/a\beta^2]$$

$$\times H_0^{(2)}\{\omega[(x-\xi)^2+\beta^2y^2]^{1/2}/a\beta^2\}\,d\xi \quad (13)$$

Consider the solution in the region where y/l is $O(1)$. This is the far field with respect to λ. The argument of $H_0^{(2)}$ is always large as $\omega\to\infty$, so that its asymptotic expansion

$$H_0^{(2)}(z)\sim(2/\pi z)^{1/2}\exp[-i(z-\pi/4)] \quad (14)$$

can be used. The integral can then be evaluated using stationary phase with the stationary point at $\xi=x-M|y|$. The unsteady perturbation velocity potential becomes

$$\phi^{(1)}=-ag(x-M|y|)\exp(-i\omega|y|/a) \quad (15)$$

It is noted that the endpoints of the integral also contribute to its asymptotic value, but if $g(x)$ is well behaved at $x=\pm l$, the contribution will be lower order in ω.

In the immediate neighborhood of the airfoil, $\phi^{(1)}$ is obtained by setting $y=0$ in Eq. (13) to yield

$$\phi^{(1)}=-\omega(2\beta)^{-1}\int_{-l}^{l} g(\xi)\exp[i\omega M(x-\xi)/a\beta^2]H_0^{(2)}$$

$$\times(\omega|x-\xi|/a\beta^2)\,d\xi \quad (16)$$

The integral has an interior critical point at $x=\xi$, which is a zero and not a stationary point. Its asymptotic value can be determined using the techniques in Bleistein and Handelsman[8] and Eq. (16) becomes

$$\phi^{(1)}=-ag(x) \quad (17)$$

which is seen to be the result in Eq. (15) with $y=0$.

The unsteady pressure distribution on the airfoil is given by Bernoulli's equation (for high frequency) as

$$p=-\rho i\omega\epsilon\phi^{(1)}\exp(i\omega t) \quad (18)$$

where ρ is the density in the stream and with the use of Eq. (17) we get

$$p=\epsilon\omega\rho iag(x)\exp(i\omega t) \quad (19)$$

Discussion

The first-order unsteady velocity potential for the pulsating airfoil at high frequency is given in Eq. (15). It is valid in a region whose extent in the transverse direction is of the order of the chordlength. It represents waves radiating from the airfoil transverse to the stream direction with speed a. Note that since the function $g(x)$ is only nonzero on the airfoil, the wave amplitude is nonzero for $|x-M|y||\leq l$. The effect of the pulsating airfoil at high frequency is therefore only felt in a region bounded by the rays $x+l=M|y|$ from the leading edge and $x-l=M|y|$ from the trailing edge. As the transverse distance from the airfoil becomes unbounded, the solution will take a different character and will decay with increasing values of y.

The unsteady pressure distribution on the airfoil surface is given in Eq. (19). It can be written in the form

$$p=\rho a[\epsilon w^{(1)}(x)\exp(i\omega t)] \quad (20)$$

which is identical to the result predicted by piston theory.

Acknowledgments

The author would like to thank R. K. Amiet of United Technologies Research Center and F. W. J. Olver of the University of Maryland for their advice on the unsteady aerodynamics and asymptotic analysis, respectively. The research was partially supported by the Minta Martin Fund for Aeronautical Research at the University of Maryland.

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Equations Governing Surface Streamlines

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Introduction

THE idealized two-dimensional flow rarely occurs in practice. Real situations always have regions of three-dimensional flow. The study of separation and attachment in three-dimensional flows focuses upon the pattern of streamlines and vortex lines on the surface. Previous discussions of the possible configurations of streamlines at separation have used a geometric basis. That is, they essentially discuss trajectories in a plane that might satisfy an arbitrary equation. This Note will present physical equations which surface streamlines must obey. These equations reveal in a general way how properties of the flowfield influence the surface streamline patterns.

We consider the incompressible viscous flow over a solid body. Streamlines on the surface are well-defined, although, since the velocity on the wall is zero, there is some reluctance to call them streamlines (see Fig. 1). They have a definite direction which may be found as the direction of the velocity vector in the limit as the wall is approached. This is also the direction of the shear stress for a Newtonian fluid, and the wall streamlines are sometimes called shear stress lines. The coincidence of the direction of the wall shear stress vector $n\cdot\tau$ (n is the unit normal perpendicular to the wall and τ is the viscous stress tensor), and the streamline direction $\tan\theta = \lim_{x_2\to 0} (v_3/v_1)$, is proved by expanding the velocity in a Taylor series about the wall:

$$v_1 = \frac{\partial v_1}{\partial x_2} x_2 + O[x_2^2], \quad v_2 = \frac{\partial^2 v_2}{\partial x_2^2} x_2^2 + O[x_2^3],$$

$$v_3 = \frac{\partial v_3}{\partial x_3} x_2 + O[x_2^2]$$

The normal component v_2 is of order x_2^2 because the continuity equation requires $\partial v_2/\partial x_2 = 0$ at the wall. Direct substitution shows the viscous stress and streamline to have the same direction.

Received July 26, 1977; revision received Nov. 18, 1977.

Index categories: Hydrodynamics; Viscous Nonboundary-Layer Flows.

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The wall also contains a set of vortex lines. They lie on the surface, i.e., $\omega_2 = 0$, as a result of the no-slip condition. By direct computation one may prove that the following relation is true at the wall:

$$\mu\omega \times n = n\cdot\tau$$

Thus, a fluid which obeys a Newtonian viscosity law will produce wall vortex lines that are perpendicular to the wall streamlines. The magnitude of the vorticity is also proportional to the wall shear stress. Lighthill's[1] discussion of wall streamlines and vortex lines emphasizes that separation in three-dimensional flows has only isolated points where the vorticity vanishes and the wall streamlines branch. The idea of a line of separation where $\omega = 0$ is an idealization for a strictly two-dimensional flow.

A differential equation which governs surface streamlines will be derived in this Note. The equation is a Poisson type with pressure and vorticity source terms. The equation should be useful in the study of three-dimensional viscous flows, three-dimensional boundary layers, and separation phenomena. Two-dimensional flows satisfy the equation in a trivial manner. The form appropriate for three-dimensional boundary layer will be presented first. Then the derivation will be generalized to arbitrary viscous flows.

Derivation for a Boundary Layer

Consider the body shown in Fig. 2 and assume the surface is regular with a continuously turning normal vector. A coordinate system is chosen so that the wall is the surface $x_2 = 0$. The flowfield vorticity is ω, and the wall vorticity is given the special symbol $\Omega(x_1,x_3)$,

$$\Omega(x_1,x_3) \equiv [\omega_1(x_1,0,x_3),0,\omega_3(x_1,0,x_3)] = \omega(x_1,0,x_3) \quad (1)$$

The wall vorticity has only two components and will be viewed as a two-dimensional vector.

The momentum equation for incompressible flow may be written as

$$\frac{\partial v}{\partial t} + \nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) = -v\times\omega - \nu\nabla\times\omega \quad (2)$$

On the wall this becomes

$$\nabla p = -\mu\nabla\times\omega \quad (3)$$

We are only interested in the normal component of this equation

$$n\cdot\nabla p = \mu(n\cdot\nabla\times\omega)$$

$$\frac{1}{h_2}\frac{\partial p}{\partial x_2} = -\frac{\mu}{h_1 h_3}\left[\frac{\partial}{\partial x_1}(h_3\omega_3) - \frac{\partial}{\partial x_3}(h_1\omega_1)\right]$$

or

$$n\cdot\nabla p = -\mu(n\cdot\hat{\nabla}\times\Omega) \quad (4)$$

where $\hat{\nabla}$ is a surface operator.

At this stage, the boundary-layer approximation $\partial p/\partial x_2 = 0$ (this is not a good assumption near separation) is used to conclude that Ω is itself an irrotational vector:

$$0 = \hat{\nabla}\times\Omega \quad (5)$$

Since Ω is irrotational, and the Stokes theorem applies in the surface (Ref. 2, p. 95), there exists a vorticity potential $\xi(x_1,x_3)$ such that

$$\Omega = \hat{\nabla}\xi \quad (6)$$